DIVERGENT SERIES AND SINGULAR POINTS OF ORDINARY DIFFERENTIAL EQUATIONS*

BY

GEORGE D. BIRKHOFF AND FREDERIC R. BAMFORTH†

Introduction

The analytic form of the solutions of the system of differential equations

$$(1)\qquad \frac{dx_1}{X_1(x_1, \cdots, x_n)} = \cdots = \frac{dx_n}{X_n(x_1, \cdots, x_n)}$$

in a neighborhood of the point $x_1 = \cdots = x_n = 0$, in which all the functions X_i are supposed to be analytic, $X_i(0, \cdots, 0)$ being zero for $i = 1, 2, \cdots, n$, has been the subject of much study. This is justified by the applications which can be made of this form to various theories in analysis and in dynamics. Dulac‡ has simplified the problem in many cases by reducing the equations (1) to simple reduced forms of which the integration can be made without difficulty. The integration of these reduced equations then furnishes the solution of the system (1) either in terms of a parameter or in the form of a system of integrals.

Let m_i be the roots of the so-called characteristic equation which, when written in determinant form, is

$$\left| \frac{\partial X_i}{\partial x_j} - m\delta_{ij} \right|_{x_1 = \cdots = x_n = 0} = 0,$$

where $\delta_{ij} = 0$ if $i \neq j$ and 1 if $i = j$, and let L_i represent the linear terms of X_i, $i = 1, 2, \cdots, n$, when these functions are expanded according to powers of $x_1, \cdots, x_n$. Three conditions which have played an important rôle in the study of the system (1) may now be written as follows: (i) if the numbers m_i are represented on a complex plane, there exists a straight line passing through the origin which is such that all the points m_i are on the same side of it; (ii) there exists a linear change of variables $x_j = x_j(y_1, \cdots, y_n)$ such that the system of differential equations

$$\frac{dx_i}{d\tau} = L_i$$

* Presented to the Society, March 29, 1929; received by the editors May 15, 1929.

† National Research Fellow.

‡ Two papers to which reference will be made in the text are the following: I: H. Dulac, Bulletin de la Société Mathématique de France, vol. 40 (1912), p. 324 et seq.; II: G. D. Birkhoff, Berlin Sitzungsberichte, 1929, pp. 171–183.

is equivalent to the system

$$\frac{dy_i}{d\tau} = m_i y_i ;$$

(iii) there exists among the numbers m_i no relationship of the form

$$m_i = p_1 m_1 + \cdots + p_n m_n, \tag{2}$$

p_j being positive integers whose sum is greater than 1. We shall refer to these conditions frequently.

On introducing a parameter t it is seen that the system (1) can be reduced to the system

$$\frac{dx_i}{dt} = X_i(x_1, \cdots, x_n) \qquad (i = 1,2, \cdots, n). \tag{3}$$

The following lemma can be readily proved* by the aid of a theorem of Poincaré's and is already in the literature:

Lemma A. *When the conditions* (i), (ii) *and* (iii) *are fulfilled, the system of differential equations* (3) *is equivalent in a neighborhood of* $x_1 = \cdots = x_n = 0$ *to the system*

$$\frac{dz_i}{dt} = m_i z_i \qquad (i = 1,2, \cdots, n), \tag{4}$$

by means of a one-to-one analytic transformation of the form

$$z_i = \phi_i(x_1, \cdots, x_n) \qquad (i = 1,2, \cdots, n), \tag{5}$$

which leaves the point $x_1 = x_2 = \cdots = x_n = 0$ *invariant.*

That this transformation is analytic is due to the fact that the formal power series solutions of the partial differential equations

$$\frac{\partial \phi_i}{\partial x_1} X_1 + \cdots + \frac{\partial \phi_i}{\partial x_n} X_n = m_i \phi_i \qquad (i = 1, \cdots, n) \tag{6}$$

converge for $|x_i| < r$, $r > 0$. But when the m_i satisfy the conditions (ii) and (iii) and not the condition (i) in the case of real variables, the power series solutions of the equations (6) may diverge. Nevertheless, as has been conjectured,† even though these series diverge there may exist transformations playing the same rôle in which the functions ϕ_i are of class C^∞ in a neighborhood of the origin and are analytic at every point of this neighborhood except

* I, p. 328.

† See II.

possibly those which lie on certain manifolds of dimensions less than n. The existence of such a transformation has been proved* for the outstanding case $n=2$ in which m_1 is greater than zero, and m_2 is less than zero. In the present paper an analogous transformation is obtained for the case $n=3$ in which the m_i are real and not all of one sign. It seems likely that the remaining important case $n=3$ not yet treated, in which m_2 and m_3 are conjugate imaginaries with real part of opposite sign to m_1, will offer essentially no new difficulties.

The case $n=3$ here treated differs essentially from the case $n=2$ in that for the former it is necessary to define two so-called invariant surfaces. The cases for $n>3$ seem to offer further difficulties. These will be discussed more fully in another paper.

Since in Section I we shall make use of another lemma which is already in the literature,† we shall quote it here in a restricted form as

LEMMA B. *When the right hand sides of the differential equations* (3) *have the form* $m_i x_i + X_{2i}$, $i=1, 2, \cdots, n$, *where* X_{2i} *have no terms of degree less than two,* $m_1, \cdots, m_p$ *are greater than zero,* $m_{p+1}, \cdots, m_n$ *are less than zero, and the condition* (iii) *is satisfied, there exists a transformation of the form*

$$y_i = x_i + f_i(x_{p+1}, \cdots, x_n) \qquad (i = 1, \cdots, p),$$
$$y_j = x_j \qquad (j = p+1, \cdots, n),$$

in which the functions f_i *are analytic in their variables in a neighborhood of* $x_{p+1} = \cdots = x_n = 0$, *which reduces the differential equations* (3) *to ones of the same type in* y_k *with the additional property that the function on the right hand side of any ith equation vanishes when* $y_1 = \cdots = y_p = 0$.

I. Preliminary analytic normalization

Consider the system of differential equations

$$\frac{dx_i}{dt} = X_i(x_1, x_2, x_3) \qquad (i = 1,2,3), \tag{7}$$

in which the X_i are real power series in the x_j, have no constant terms, and converge for $|x_j| \leqq r_1$ where $r_1 > 0$. The coefficients in the X_i are constants and not functions of the parameter t. Let m_1, m_2 and m_3 be the roots of the determinant equation

$$\left| \frac{\partial X_i}{\partial x_j} - m\delta_{ij} \right|_{x_1=x_2=x_3=0} = 0.$$

* Loc. cit. II.

† I, p. 359 et seq.

It will be assumed that m_1 and m_2 are greater than zero, m_3 is less than zero, and that the conditions (ii) and (iii) mentioned in the Introduction are satisfied.

The object of this paper is to show that the system of differential equations (7) is equivalent to the system

$$\frac{dz_i^*}{dt} = m_i z_i^* \qquad (i = 1,2,3), \tag{8}$$

by means of a one-to-one transformation of the type

$$z_i^* = \phi_i(x_1, x_2, x_3) \tag{9}$$

where the functions on the right involve the displayed arguments only, are zero at the origin, $x_1 = x_2 = x_3 = 0$, are continuous together with all their partial derivatives within a certain neighborhood of the origin, are analytic at any point of this neighborhood which does not lie on any one of a certain set of three surfaces which pass through the origin, and have certain further analyticity properties. If m_3 were positive instead of negative and the conditions (ii) and (iii) were still satisfied, then the transformation (9) might be chosen analytic in a neighborhood of the origin as Lemma A shows. However, when m_3 is negative, a transformation of the type just mentioned is the closest approach to an analytic transformation which we have discovered that will reduce the equations (7) to the equations (8). The equivalence of these two systems will only be shown to exist for a certain small neighborhood of the origin.

By an "invariant surface" will be meant a surface in the space of the dependent variables which is made up of integral curves of the differential equations (7) and contains the origin as an interior point; by an "invariant curve" will be meant an integral curve of (7) which passes through the origin. Now the classical theory of the system of differential equations (7) tells us that there exists an analytic invariant surface which contains two analytic invariant curves, and that there exists a third analytic invariant curve which does not lie in this surface. The first sequence of transformations which will be used will reduce the equations (7) to a special form in which these three analytic invariant curves are the three axes in the space of the dependent variables and the analytic invariant surface is one of the coordinate planes.

On account of the assumption that the condition (ii) is satisfied, we may assume that the right hand members of the differential equations (7) have the forms

$$X_i = m_i x_i + X_{2i}(x_1, x_2, x_3) \qquad (i = 1,2,3), \tag{10}$$

in which the X_{2i} have no terms of less than the second degree in their arguments when they are expanded as power series about the origin.

Now let us examine the differential equations (7) whose right hand members have the forms given in (10) in the light of Lemmas A and B of the Introduction. Since we may perform the two analytic transformations which Lemma B assures us exist, the first being obtained when we consider m_1, m_2 as the $m_1, \cdots, m_p$ of the lemma, and the second when we change t to $-t$ and consider m_3 as the $m_1, \cdots, m_p$ of the lemma, we may assume that the X_{2i} of (10) have the property that

$$X_{21}(0,0,x_3) = X_{22}(0,0,x_3) = X_{23}(x_1,x_2,0) = 0. \tag{11}$$

The performing of these transformations moves the third analytic invariant curve and the analytic invariant surface, which have been mentioned before, into the $x_1 = x_2 = 0$ axis and the $x_3 = 0$ plane, respectively.

We wish now to move the two analytic invariant curves in the $x_3 = 0$ plane into the coördinate axes of that plane as well as make further simplifications. Since we wish to work in the $x_3 = 0$ plane, let us set $x_3 = 0$ which is seen to satisfy the third differential equation of (7) on account of the equations (11). The other two equations of (7) then take the form

$$\frac{dx_1}{dt} = m_1 x_1 + X_{21}(x_1, x_2, 0), \quad \frac{dx_2}{dt} = m_2 x_2 + X_{22}(x_1, x_2, 0).$$

In as much as these equations satisfy the hypotheses of Lemma A, and we may perform the transformation which that lemma says exists, we may assume that

$$X_{21}(x_1, x_2, 0) = X_{22}(x_1, x_2, 0) = 0$$

which implies that the analytic invariant curves in the $x_3 = 0$ plane are the coördinate axes.

Since, on account of the equations (11), which are still true after the transformation just mentioned has been performed, $x_1 = 0$, $x_2 = 0$ satisfies the first two equations of (7), we may assume that $X_{23}(0, 0, x_3) = 0$, because if it were not so we could make use of an analytic transformation of the form

$$y_1 = x_1, \quad y_2 = x_2, \quad y_3 = x_3 + f(x_3),$$

whose existence is assured by Lemma A, to make $Y_{23}(0, 0, y_3) = 0$.

As a result of this discussion we may state that *without loss of generality, the differential equations* (7) *may be assumed to have the form*

$$\frac{dx_1}{dt} = x_1(m_1 + x_3P_1) + x_2x_3Q_1,$$
$$\frac{dx_2}{dt} = x_2(m_2 + x_3P_2) + x_1x_3Q_2, \tag{12}$$
$$\frac{dx_3}{dt} = x_3(m_3 + x_1P_3 + x_2P_4),$$

where the P_i and Q_i are analytic functions of their arguments x_j in a neighborhood of the origin, for $|x_i| \leqq r$, $r>0$, $i=1, 2, 3$, Q_1 is a function of x_2 and x_3 only, and Q_2 is a function of x_1 and x_3 only.

II. Convergence properties of formal series

Let us recall that the aim of the present paper is to find three functions $\phi_i(x_1, x_2, x_3)$ such that, when the x_j are replaced by an arbitrary solution $x_j(t)$ of the differential equations (12), $z_i^* = \phi_i[x_1(t), x_2(t), x_3(t)]$ satisfy the differential equations (8). Since all three of these functions can be found in exactly the same way we shall confine our attention for the present to ϕ_1. Evidently ϕ_1 must be a solution of the partial differential equation

$$\frac{\partial\phi_1}{\partial x_1}[x_1(m_1 + x_3P_1) + x_2x_3Q_1] + \frac{\partial\phi_1}{\partial x_2}[x_2(m_2 + x_3P_2) + x_1x_3Q_2] + \frac{\partial\phi_1}{\partial x_3}[x_3(m_3 + x_1P_3 + x_2P_4)] = m_1\phi_1. \tag{13}$$

Equation (13) has a formal power series solution whose only linear term is ax_1, where a is an arbitrary constant. This power series solution is unique if the coefficient of x_1 is chosen as 1. Let us assume that this has been done. Furthermore, every other term has a factor x_1x_3 or x_2x_3 as is readily seen by alternately setting $x_1 = x_2 = 0$ and $x_3 = 0$ in (13) and solving for ϕ_1. Let this formal series be

$$\phi_1 \sim x_1 + \cdots + \frac{a_{mnp}}{m!n!p!} x_1^m x_2^n x_3^p + \cdots, \tag{14}$$

which obviously can be written in either of the forms

$$\phi_1 \sim \alpha_0(x_1, x_2) + \alpha_1(x_1, x_2)x_3 + \cdots + \alpha_p(x_1, x_2)x_3^p + \cdots, \tag{15}$$

or

(16) $$\phi_1 \sim \alpha_{00}(x_3) + \alpha_{10}(x_3)x_1 + \cdots + \alpha_{mn}(x_3)x_1^m x_2^n + \cdots .$$

We shall now prove

LEMMA 1. *The series* α_p *of* (15) *and the series* α_{mn} *of* (16) *all converge for* $|x_1|$, $|x_2|<r$ *and for* $|x_3|<r$ *respectively.*

Substitute the series (15) into the equation (13) and equate the coefficients of like powers of x_3. Evidently $\alpha_0=x_1$, and α_1 satisfies the equation

(17) $$\frac{\partial\alpha_1}{\partial x_1}x_1m_1 + \frac{\partial\alpha_1}{\partial x_2}x_2m_2 + \alpha_1(m_3 - m_1 + x_1P_{30} + x_2P_{40}) = \Phi_1,$$

where $P_{i0}=P_i(x_1, x_2, 0)$ and Φ_1 is linear in α_0 with coefficients which are analytic in x_1, x_2 for $|x_1|$, $|x_2|\leqq r$, and are zero for $x_1=x_2=0$.

Set

(18) $$\alpha_1 = \beta_0(x_2) + \cdots + \beta_s(x_2)x_1^s + \cdots ,$$

where the β_i are formal power series in x_2. Then β_0 formally satisfies an equation of the form

$$\frac{d\beta_0}{dx_2}x_2m_2 + \beta_0(m_3 - m_1 + x_2P_{400}) = \Phi_{10},$$

where P_{400} and Φ_{10} are analytic in x_2 for $|x_2|\leqq r$. On account of the condition (iii) of the Introduction being satisfied, it readily follows that β_0 is analytic† in x_2 for $|x_2|\leqq r$. In a similar manner it can be shown that all the β_i are analytic in x_2 for $|x_2|\leqq r$, and if α_1 be arranged according to ascending powers of x_2 it can be shown in the same manner that the coefficients are analytic in x_1 for $|x_1|\leqq r$.

Now define

$$B_s = \beta_0 + \beta_1x_1 + \cdots + \beta_{s-1}x_1^{s-1}$$

and $\gamma_1=\alpha_1-B_s$. Evidently γ_1 has a factor x_1^s, and hence, writing $\gamma_1=x_1^s\xi_1$, we see that ξ_1 will satisfy an equation of the form

$$\frac{\partial\xi_1}{\partial x_1}x_1m_1 + \frac{\partial\xi_1}{\partial x_2}x_2m_2 + \xi_1(m_3 - m_1 + sm_1 + x_1P_{30} + x_2P_{40}) = \Phi_{11},$$

where Φ_{11} is analytic in x_1, x_2 for $|x_1|$, $|x_2|\leqq r$. Let s be large enough so that $m_3+m_1(s-1)$ is positive and define m_4 as equal to this quantity. Let m be a number that is greater than zero and less than the smallest of m_1, m_2 and m_4. Then

$$|\,pm_1 + qm_2 + m_4\,| > m(p + q + 1)$$

† Cf. II, pp. 177–178.

for any pair of positive integers p, q. Choose $M>0$ and such that x_1P_{30} and x_2P_{40} are dominated by

$$\frac{M}{\left(1-\frac{x_1}{r}\right)\left(1-\frac{x_2}{r}\right)}-M, \tag{19}$$

and Φ_{11} is dominated by the first term of (19) only. Then the power series ξ_1 is dominated by the power series solution of

$$m\left(x_1\frac{\partial f}{\partial x_1}+x_2\frac{\partial f}{\partial x_2}\right)=f\left[-m+\frac{M}{\left(1-\frac{x_1}{r}\right)\left(1-\frac{x_2}{r}\right)}-M\right]+\frac{M}{\left(1-\frac{x_1}{r}\right)\left(1-\frac{x_2}{r}\right)}\cdot \tag{20}$$

On account of the symmetry in (20) with respect to x_1 and x_2 the power series solution of (20) converges for $|x_1|$, $|x_2|<R$, where R is the radius of convergence of the power series solution of the differential equation

$$mx_1\frac{dF}{dx_1}=F\left[-m+\frac{M}{\left(1-\frac{x_1}{r}\right)^2}-M\right]+\frac{M}{\left(1-\frac{x_1}{r}\right)^2}\cdot$$

From this it follows that $R=r$. Hence α_1 is analytic in x_1, x_2 for $|x_1|$, $|x_2|<r$.

The proof of the lemma for the other α_i follows readily by induction since every α_i satisfies an equation of the form (17) where α_1 is replaced by α_i, m_3 by im_3, and Φ_1 by Φ_i, where Φ_i is a polynomial in $\alpha_0, \cdots, \alpha_{i-1}$, with coefficients which are analytic in x_1, x_2 for $|x_1|$, $|x_2|\leqq r$.

The α_{mn} can be proved to be analytic in the same way that the β_i of (18) are proved analytic. This completes the proof of Lemma 1.

Similarly, we may prove that when the series corresponding to ϕ_2 and ϕ_3 are arranged as power series in x_3 the coefficients are analytic in x_1, x_2 for $|x_1|$, $|x_2|<r$, and when arranged as power series in x_1, x_2 the coefficients are analytic in x_3 for $|x_3|<r$. The only linear term in ϕ_2 is the one involving x_2, and every other term contains as a factor x_1x_3 or x_2x_3. The coefficient of the linear term is arbitrary and when it is chosen as 1 the series for ϕ_2 is unique. Let us suppose that this choice has been made. The series for ϕ_3 has properties similar to those of ϕ_1 and ϕ_2 with the additional one that x_3 is a factor of every term in the series. We shall choose the coefficient of

the linear term as 1 and this choice uniquely determines all the others.

If the formal series for ϕ_i, $i=1$, 2, or 3, should converge for $|x_i|<r_0$, $r_0>0$, $i=1, 2, 3$, it can evidently be used to reduce the ith differential equation of (12) immediately to the normal form $dz_i^*/dt=m_iz_i^*$.

Even though the series for the ϕ_i should diverge, it will be found that they can be "fitted" by functions of class C^∞ in a manner described in

LEMMA 2. *Let the formal series*

$$S(x_1, x_2, x_3) = \sum_{m,n,p=0}^{\infty} \frac{a_{mnp}}{m!n!p!} x_1^m x_2^n x_3^p,$$

in which all the a_{mnp} are real, have the property that when it is arranged as a power series in x_3, i.e.,

$$(21) \qquad S = \alpha_0(x_1, x_2) + \alpha_1(x_1, x_2)x_3 + \cdots + \alpha_p(x_1, x_2)x_3^p/p! + \cdots,$$

the α_p are analytic in the complex variables x_1, x_2 for $|x_1|$, $|x_2|\leqq r$, $r>0$, and when it is arranged as a power series in x_1, x_2, i.e.,

$$(22) \qquad S = \alpha_{00}(x_3) + \alpha_{10}(x_3)x_1 + \cdots + \alpha_{mn}(x_3)x_1^m x_2^n/(m!n!) + \cdots,$$

the α_{mn} are analytic in the complex variable x_3 for $|x_3|\leqq r$. Then there exists a real function $F(x_1, x_2, x_3)$ of the real variables x_1, x_2, x_3 which is continuous together with all its partial derivatives for $|x_1|$, $|x_2|$, $|x_3|\leqq r$, is analytic for $x_1\neq 0$, $x_2\neq 0$, $x_3\neq 0$, satisfies the equations

$$(23) \qquad \left.\frac{\partial^{m+n}F}{\partial x_1^m \partial x_2^n}\right|_{x_1=x_2=0} = \alpha_{mn}(x_3), \quad \left.\frac{\partial^p F}{\partial x_3^p}\right|_{x_3=0} = \alpha_p(x_1, x_2) \ (m, n, p = 0, 1, \cdots),$$

and has the property that it and all its partial derivatives are analytic in x_i and x_j for $|x_i|$, $|x_j|\leqq r$, $x_i\neq 0$, $x_j\neq 0$, $x_k=0$, $i\neq j$, $i\neq k$, $j\neq k$, and are analytic in x_i for $|x_i|\leqq r$, $x_j=x_k=0$, $i, j, k=1, 2, 3$.

Consider the series†

$$(24) \qquad F(x_1, x_2, x_3) = \sum_{m,n,p=0}^{\infty} \frac{a_{mnp}}{m!n!p!} x_1^m x_2^n x_3^p (1 - e^{-1/B_{mnp}}),$$

where $i=1$ if $m\geqq n$, $i=2$ if $m<n$, $b_{mnp}=1+|a_{mnp}|$, $B_{mnp}=b_{mnp}x_3^2x_i^2$. At first we shall consider this series only in the six-dimensional complex region defined by

$$(25) \qquad |x_j| \leqq r, \quad |\arg x_j| \leqq \theta, \quad 0<\theta<\pi/8 \qquad (j = 1, 2, 3).$$

In this region $|\arg x_3^2x_i^2|$ does not exceed $\pi/2$, so that the exponential

† Cf. II, p. 173.

terms $e^{-1/B_{mnp}}$ in the series have exponents $-1/B_{mnp}$ with negative real parts, and so are less than one in absolute value.

Moreover, we find for $|B_{mnp}| \geqq 1$,

$$| a_{mnp}(1 - e^{-1/B_{mnp}}) | = | a_{mnp} | \cdot \left| \frac{1}{B_{mnp}} - \frac{1}{2!B_{mnp}^2} + \cdots \right|$$
$$\leqq | a_{mnp} | \cdot (e - 1)/ | B_{mnp} | \leqq 2/[| x_3 |^2 | x_i |^2].$$

But if, on the contrary, $|B_{mnp}| < 1$, then, by the definition of B_{mnp}, $|a_{mnp}| \leqq 1/[|x_3|^2 \ |x_i|^2]$. Since the quantity in the parentheses in the preceding inequality is obviously less than two in absolute value, we conclude that the inequality holds in this case too and so without restriction on B_{mnp}. Thus the given series is dominated by the series

$$\sum_{m,n,p=0}^{\infty} \frac{2}{m!n!p!} | x_1 |^{m_1} | x_2 |^{n_1} | x_3 |^{p-2}, \tag{26}$$

where

$$\begin{aligned} m_1 &= m - 2 \text{ if } m \geqq n \text{ and } m \text{ if } m < n, \\ n_1 &= n - 2 \text{ if } m < n \text{ and } n \text{ if } m \geqq n. \end{aligned} \tag{27}$$

But it follows from the hypotheses concerning the α_p that that part of the series (24) for which $p \leqq 3$ converges uniformly in the region (25); and it follows from the hypotheses concerning the α_{mn} that that part of the series (24) which gives rise to a negative m_1 or to a negative n_1 converges uniformly in the same region. Hence the series (24) converges uniformly in the region (25), and an examination of this series will readily show that it is analytic except for $x_1 = 0$, $x_2 = 0$, or for $x_3 = 0$, and that

$$F(x_1, x_2, 0) = \alpha_0, \quad F(0, 0, x_3) = \alpha_{00}.$$

In a similar manner, all the series obtained from F by partial differentiation can be shown to have the same properties of analyticity and continuity as has F in the region defined by (25). This is on account of the fact that the α_p of (21) and the α_{mn} of (22) are analytic for $|x_1|$, $|x_2|$, $|x_3| \leqq r$ and that for a finite number of differentiations of (24) only a finite number of negative powers of x_1 and of x_2 are introduced into the dominating series which correspond to these derivative series in the same way that the series (26) does to F. We shall consider only $\partial F/\partial x_3$,

$$\frac{\partial F}{\partial x_3} = \sum_{m,n,p-1=0}^{\infty} \frac{a_{mnp}}{m!n!(p-1)!} x_1^m x_2^n x_3^{p-1}(1 - e^{-1/B_{mnp}})$$
$$+ \sum_{m,n,p=0}^{\infty} \frac{a_{mnp}}{m!n!p!} x_1^m x_2^n x_3^p \left(\frac{-2e^{-1/B_{mnp}}}{B_{mnp} x_3} \right).$$

The first of these series is evidently dominated by

$$\sum_{m,n,p-1=0}^{\infty} \frac{2\,|\,x_1\,|^{m_1}\,|\,x_2\,|^{n_1}\,|\,x_3\,|^{p-3}}{m!n!(p-1)!},$$

and the second by

$$\sum_{m,n,p=0}^{\infty} \frac{2\,|\,x_1\,|^{m_1}\,|\,x_2\,|^{n_1}\,|\,x_3\,|^{p-3}}{m!n!p!}.$$

From the remarks made above concerning α_p and α_{mn}, etc., it readily follows that $\partial F/\partial x_3$ has the same properties of continuity and analyticity as has F itself. Furthermore,

$$\left.\frac{\partial F}{\partial x_3}\right|_{x_3=0} = \alpha_1.$$

$F(x_1, x_2, x_3)$ is thus a real function in the octant $x_1 \geqq 0$, $x_2 \geqq 0$, $x_3 \geqq 0$, continuous together with all its partial derivatives in this octant, analytic for $x_1>0$, $x_2>0$, $x_3>0$, and satisfies the equations (23). Similarly F is seen to have the same properties in each of the other octants and the eight functions thus defined evidently unite to form a function $F(x_1, x_2, x_3)$ defined by the series (24) and having the properties stated in the lemma down to and including those relative to the equations (23). To prove the last statements made in the lemma concerning the function F and its partial derivatives we need merely use dominating series in a manner wholly analogous to that in which they were used above.

III. Final analytic normalization

The problem in hand is more readily discussed if the functions Q_1 and Q_2 appearing in the differential equations (12) have the factors x_2x_3 and x_1x_3, respectively. We shall now show that the assumption that such is the case can be made without the introduction of any further hypotheses than have already been made. To this end let us consider the transformation

$$(28) \qquad y_i = s_i(x_1, x_2, x_3) \qquad (i = 1, 2, 3),$$

where the series $s_i(x_1, x_2, x_3)$ consist of all the terms of the formal series for ϕ_i, $i=1, 2, 3$, respectively, involving $x_1^m x_2^n$, $m+n=0, 1, \cdots, \lambda+1$, all the terms involving x_3^p, $p=0, 1, \cdots, \lambda+1$, and no others. For the present λ will be considered as any integer greater than 1 but will be determined in a later section as depending on the m_i. The properties of convergence of the coefficients of the formal series for the ϕ_i when these series are arranged according to powers of x_3 or according to powers of x_1 and x_2 imply that the series s_i converge for $|x_j|<r$.

From the definition of the s_i it follows that

$$\frac{\partial s_i}{\partial x_1}[x_1(m_1 + x_3P_1) + x_2x_3Q_1] + \frac{\partial s_i}{\partial x_2}[x_2(m_2 + x_3P_2) + x_1x_2Q_2]$$
$$+ \frac{\partial s_i}{\partial x_3}x_3(m_3 + x_1P_3 + x_2P_4) - m_is_i$$

define functions $R_i(x_1, x_2, x_3)$ which have no terms of degree less than $\lambda+2$ in x_3 and less than $\lambda+2$ in x_1 and x_2 together. Since the only linear term in s_i is x_i, $i=1, 2, 3$, respectively, when we solve (28) for $x_i=t_i(y_1, y_2, y_3)$ the only linear term in t_i is y_i, $i=1, 2, 3$, respectively. Hence the R_i when expressed as functions of y_j have no terms of degree less than $\lambda+2$ in y_3 and less than $\lambda+2$ in y_1 and y_2 together. Thus, when x_i are replaced by an arbitrary solution $x_i(t)$ of (12), the corresponding y_i satisfy the system of differential equations

$$\frac{dy_i}{dt} = m_iy_i + R_i \qquad (i = 1,2,3), \tag{29}$$

where the arguments of the R_i are the y_j. But every term in the formal series for ϕ_1, except the linear term, contains x_1x_3 or x_2x_3 as a factor, and corresponding statements are true concerning the formal series for the ϕ_2 and ϕ_3. This implies that when $x_3=0$, then $y_3=0$, $x_1=y_1$ and $x_2=y_2$, and when $x_1=x_2=0$, then $y_1=y_2=0$ and $x_3=y_3$. Thus the transformation (28) will preserve the form of the differential equations (12) and hence the form of the system (29) is the same as that of the system (12) when the x are replaced by the y_i, $i=1, 2, 3$. Furthermore, on account of the properties of the R_i we see that the new Q_1 will have $y_2^\lambda y_3^\lambda$ as a factor and the new Q_2 will have $y_1^\lambda y_3^\lambda$ as a factor. Hence we are not losing in generality when we assume that the differential equations under discussion have the form

$$\begin{aligned} \frac{dx_1}{dt} &= x_1(m_1 + x_3P_1) + x_2^2 x_3^2Q_1, \\ \frac{dx_2}{dt} &= x_2(m_2 + x_3P_2) + x_1^2 x_3^2Q_2, \\ \frac{dx_3}{dt} &= x_3(m_3 + x_1P_3 + x_2P_4), \end{aligned} \tag{30}$$

where the P_i and Q_i are analytic in x_j for $|x_j| \leq r$, $r>0$, $j=1, 2, 3$, Q_1 is a function of x_2 and x_3 only, and Q_2 is a function of x_1 and x_3 only. Furthermore, every P_i has $x_3^{\lambda+1}$ as a factor, and when expanded as a power series in

x_1 and x_2 contains no term of degree less than $\lambda+1$ in x_1 and x_2 together; Q_1 contains $x_2^\lambda x_3^\lambda$ as a factor, and Q_2 contains $x_1^\lambda x_3^\lambda$ as a factor. These facts will be used in the next section. The partial differential equation (13) and the corresponding ones in ϕ_2 and ϕ_3 will now be replaced by the system

$$\begin{aligned}(31)\qquad &\frac{\partial\phi_i}{\partial x_1}[x_1(m_1 + x_3P_1) + x_2^2x_3^2Q_1] + \frac{\partial\phi_i}{\partial x_2}[x_2(m_2 + x_3P_2) + x_1^2x_3^2Q_2] \\ &\qquad + \frac{\partial\phi_i}{\partial x_3}[x_3(m_3 + x_1P_3 + x_2P_4)] = m_i\phi_i \qquad (i = 1,2,3).\end{aligned}$$

IV. Construction of auxiliary invariant surfaces

It happens that if $Q_1 = Q_2 = 0$ we can fairly readily obtain solutions of this system (31) of class C^∞ in a neighborhood of the origin, and thus solve our problem. Hence we shall try to find a transformation of the form

$$(32)\qquad x_i = x_i(y_1, y_2, y_3) \qquad (i = 1,2,3),$$

which will reduce this system of partial differential equations to an equivalent system in which the coefficient of $\partial\phi_i/\partial y_j$ has y_j as a factor, $i, j = 1, 2, 3$. Evidently we may choose $x_3 = y_3$ as the last equation in this transformation. Now if the system of partial differential equations when expressed in terms of the y_j have this additional property, the transformation (32) will take the integral curves of the system of differential equations (30) into the integral curves of a system of differential equations of the form

$$\frac{dy_i}{dt} = y_i(m_i + T_i) \qquad (i = 1,2,3).$$

Hence our search for such a transformation (32) is connected with the search for two invariant surfaces of the system of differential equations (30) other than the known one $x_3 = 0$. In this connection we shall prove

THEOREM 1. *There exists a function $f_1(x_2, x_3)$ which is of class C^∞ in a neighborhood of $x_2 = x_3 = 0$, is analytic in x_2, x_3 for $x_2 \neq 0$, $x_3 \neq 0$, is, together with all its partial derivatives, analytic in either variable when the other variable is zero, and is such that*

$$x_1 = x_2^2x_3^2f_1(x_2, x_3)$$

is an invariant surface. There also exists another function $g_1(x_1, x_3)$ with properties analogous to those of $f_1(x_2, x_3)$ such that

$$x_2 = x_1^2x_3^2g_1(x_1, x_3)$$

is another invariant surface.

Since the method to be used for the determination of one of these invariant surfaces can be used for the determination of the other, we shall confine our attention to that surface which will be transformed into the $y_1=0$ plane. To this end consider the transformation

$$y_1 = x_1 - x_2^2 x_3^2 f_1(x_2, x_3), \quad y_2 = x_2, \quad y_3 = x_3, \tag{33}$$

where the function f_1 is to be so determined that when the x_i are replaced by an arbitrary solution $x_i(t)$ of (30) the $y_1(t)$ so defined will satisfy a differential equation of the form

$$\frac{dy_1}{dt} = y_1(m_1 + T_1), \tag{34}$$

in which T_1 is a function of the y_i of class C^∞ and $T_1(0,0,0)=0$. On differentiating the first equation of (33) with respect to t, replacing the derivatives dx_i/dt by their values given in the equations (30) and x_1 by its value given in (33), and noting that $y_1=0$ is to be a solution of the resulting differential equation on account of the required form (34), we obtain the result that $f_1(x_2, x_3)$ is a solution of the partial differential equation

$$\begin{aligned}\frac{\partial f_1}{\partial x_2} x_2(m_2 + x_3R_1) + \frac{\partial f_1}{\partial x_3} x_3(m_3 + x_2R_2)& \\ = f_1(m_4 - 2x_2R_2 + x_3R_3) + Q_1&\end{aligned} \tag{35}$$

in which

$$m_4 = m_1 - 2m_2 - 2m_3, \quad R_1 = P_2 + x_2^3 x_3^5 f_1^2 Q_2,$$

$$R_2 = x_2 x_3^2 f_1 P_3 + P_4, \qquad R_3 = P_1 - 2R_1,$$

where the P_i and the Q_2 are the same functions of $x_2^2 x_3^2 f_1$, x_2, x_3 as they were of x_1, x_2, x_3 in (30). From the fact that every P_i has $x_3^{\lambda+1}$ as a factor and when expanded as a power series in x_1 and x_2 contains no term of degree less than $\lambda+1$ in x_1 and x_2 together, and from the form of the R_i it follows that every R_i has $x_2^{\lambda+1}x_3^{\lambda+1}$ as a factor. If $f_1(x_2, x_3)$ is any solution of the equation (35) which is of class C^∞ in a neighborhood of the origin, the function y_1 so defined by the first equation in (33) will satisfy an equation of the form (34) when the x_i are replaced by an arbitrary solution $x_i(t)$ of the differential equations (30). This fact is an immediate consequence of the manner in which the partial differential equation (35) has been derived.

We shall now prove

LEMMA 3. *There exists a solution $f_1(x_2, x_3)$ of the partial differential equation* (35) *which is of class C^∞ in a neighborhood of the origin $x_2=x_3=0$ and is analytic at any interior point of this neighborhood for which $x_2\neq 0$, $x_3\neq 0$. Furthermore, this solution and all its partial derivatives are analytic in either variable in a neighborhood of $x_i=0$, $i=2$ or 3 as the case may be, when the other variable is zero.*

From what has gone before, Theorem 1 will be obvious when Lemma 3 has been proved.

The first thing to be observed is that there exists a unique formal power series solution of the partial differential equation (35) which may be written in any one of the three forms

$$
\begin{aligned}
f_1 &\sim a_{00} + \cdots + \frac{a_{mn}}{m!n!} x_2^m x_3^n + \cdots \\
&= \alpha_0(x_2) + \cdots + \alpha_m(x_2) x_3^m + \cdots \qquad (36)\\
&= \beta_0(x_3) + \cdots + \beta_n(x_3) x_2^n + \cdots .
\end{aligned}
$$

Since every R_i and Q_1 has $x_2^\lambda x_3^\lambda$ as a factor, this formal series for f_1 also has $x_2^\lambda x_3^\lambda$ as a factor. If this series does not diverge for every pair of values $x_2\neq 0$, $x_3\neq 0$, our lemma is obviously proved, so that we shall confine our attention in the following discussion to the contrary case. By making use of the methods employed in the proof of Lemma 1 it can be shown that all the $\alpha_m(x_2)$ are analytic in x_2 for $|x_2|<r$ and that all the $\beta_n(x_3)$ are analytic in x_3 for $|x_3|<r$. This follows from the fact that wherever f_1 appears in P_i or in Q_2 it is multiplied by $x_2^2 x_3^2$. Now define a function $g(x_2, x_3)$ which corresponds to the formal series for f_1 in the same way that the function F corresponds to the series S in Lemma 2, it being supposed for this application of this lemma that no x_1 appears in the series S and that that part of the function F which is independent of x_2 is replaced by the corresponding analytic series which is a part of S. On making the transformation $f_1=f+g$ we see that f_1 satisfies the differential equation (35) if and only if f is a solution of the partial differential equation

$$
\frac{\partial f}{\partial x_2} x_2(m_2 + x_3 S_1) + \frac{\partial f}{\partial x_3} x_3(m_3 + x_2 S_2) = f(m_4 + S_3) + S_4, \qquad (37)
$$

in which the S_i are well defined functions of f, x_2, x_3, $S_3(0, 0, 0)$ being zero and S_4 being independent of f. These functions are given explicitly by the following equations (38) and have the properties that they are of class C^∞ in a neighborhood of $f=x_2=x_3=0$, are analytic in f in a neighborhood of $f=0$ for x_2 and x_3 in a neighborhood of $x_2=x_3=0$, and are analytic in x_2, x_3 in a neighborhood of $x_2=x_3=0$ for $x_2\neq 0$, $x_3\neq 0$ and f in a neighborhood of $f=0$. All these properties are immediately implied by the equations

$$
\begin{aligned}
S_1(f,x_2,x_3) &= R_1(f+g,x_2,x_3),\quad S_2(f,x_2,x_3) = R_2(f+g,x_2,x_3),\\
S_3(f,x_2,x_3) &= -2x_2R_2 + x_3R_3 - \frac{1}{f}\Big\{\frac{\partial g}{\partial x_2}x_2x_3(R_1-R_{10})\\
&\quad + \frac{\partial g}{\partial x_3}x_2x_3(R_2-R_{20}) - g[-2x_2(R_2-R_{20})\\
&\quad + x_3(R_3-R_{30})]\Big\},\\
S_4(f,x_2,x_3) &= Q_1 + g(m_4 - 2x_2R_{20} + x_3R_{30}) - \frac{\partial g}{\partial x_2}x_2(m_2+x_3R_{10})\\
&\quad - \frac{\partial g}{\partial x_3}x_3(m_3 + x_2R_{20}),
\end{aligned}
\tag{38}
$$

where the arguments of the R_i are $f+g$, x_2, x_3 and $R_{i0}=R_i(0+g,\ x_2,\ x_3)$. Evidently every S_i has $x_2^\lambda x_3^\lambda$ as a factor. This fact will be made use of later on. From the definition of $g(x_2, x_3)$ it follows that every partial derivative of S_4 is zero when either x_2 or x_3 is zero. Hence for every pair of positive integers p, q a function S_{pq} can be defined which has the same properties of continuity and analyticity that S_4 itself has and satisfies the equation

$$S_4 = x_2^p x_3^q S_{pq}. \tag{39}$$

In other words, S_4 admits of the factor $x_2^p x_3^q$ for arbitrary p and q. This fact can readily be established by taking the factors x_2 and x_3 out of S_4 one at a time, e.g., for $x_2\neq 0$ the function S_4/x_2 has all the properties of continuity and of analyticity that S_4 itself has. The limits as $x_2\to 0$ of S_4/x_2 and all its partial derivatives exist and are zero. Hence S_4/x_2 defines S_{10}.

Now let us consider the system of ordinary differential equations

$$
\begin{aligned}
\frac{df}{dt} &= f(m_4+S_3)+S_4,\\
\frac{dx_2}{dt} &= x_2(m_2+x_3S_1),\\
\frac{dx_3}{dt} &= x_3(m_3+x_2S_2).
\end{aligned}
\tag{40}
$$

If we can determine a surface $f=f(x_2, x_3)$ in the f, x_2, x_3 space which is made up of integral curves of the system of ordinary differential equations (40), such that $f(x_2, x_3)$ is of class C^∞ in a neighborhood of $x_2=x_3=0$, and is analytic at any interior point of this neighborhood for which $x_2\neq 0$, $x_3\neq 0$, the function $f(x_2, x_3)$ so defined will be a solution of the partial differential equation (37) and will define by means of the equation $f_1=f+g$ a function $f_1(x_2, x_3)$ which will be a solution of the partial differential equation (35). Thus if we can find such an $f(x_2, x_3)$ our lemma will be proved.

Suppose for the sake of definiteness that m_4 is greater than zero. We shall limit ourselves to a closed region R about $f=x_2=x_3=0$ in which $|x_i|<1$, $i=2, 3$, m_4+S_3 and $m_2+x_3S_1$ are greater than zero, and $m_3+x_2S_2$ is less than zero. We shall suppose that R is further limited so that for (f, x_2, x_3) any point in this region the functions S_i are analytic in f, of class C^∞ in x_2, x_3, and are analytic in x_2, x_3 if $x_2\neq 0$, $x_3\neq 0$. That this is possible follows from the properties of the S_i stated above. The following notation will be adopted for this region R:

$$\tag{41} \begin{aligned} M_1 &= \max (m_4 + S_3), & M_2 &= \max (m_2 + x_3S_1), & M_3 &= \max (m_3 + x_2S_2), \\ N_1 &= \min (m_4 + S_3), & N_2 &= \min (m_2 + x_3S_1), & N_3 &= \min (m_3 + x_2S_2). \end{aligned}$$

Now consider the surface defined by the integral curves of the differential equations (40) which pass through the line

$$\tag{42} f = 0,\ x_2 = x_3 = \tau,\ \tau \geqq 0,$$

and suppose that $t=0$ on this line. The equations of this surface may be written as

$$\tag{43} f = f_0(t,\tau),\ x_2 = x_2(t,\tau),\ x_3 = x_3(t,\tau),$$

where the functions appearing on the right hand sides of these equations, on account of the analytic properties of the S_i, are analytic in their arguments so long as $\tau\neq 0$. Until further notice, it will be understood that the discussion from now on is relative to that quadrant of R for which x_2 and x_3 are positive. We wish first of all to show that there is an open region R_0 of the x_2, x_3 space which is bounded by the axes $x_2=0$, $x_3=0$ and by the arc of a circle, having its center at the origin and its radius different from zero, and is of such a nature that, when (x_2, x_3) is any point in it, the last two equations of (43) can be solved for t and τ as single-valued functions of x_2 and x_3. For such a region these functions $t=t(x_2, x_3)$ and $\tau=\tau(x_2, x_3)$ will evidently be analytic, and when they are substituted in the first equation of (43) we shall have $f=f(x_2, x_3)$ where $f(x_2, x_3)$ is an analytic function of its arguments for (x_2, x_3) in R_0.

To this end let us find how the t of any point of the surface (43) depends on the corresponding τ. From the second equation of (40) we evidently have

$$\log x_2 - \log \tau = \int_0^t (m_2 + x_3S_1)dt.$$

Since $x_2<1$ on account of the hypotheses concerning the region R, for $t\geqq 0$ we may write $-\log\tau\geqq N_2t$ where N_2 is defined in (41). A corresponding inequality exists for $t\leqq 0$, i.e., $-\log\tau\geqq M_3t$, so that we know that

$$|t| \leqq n|\log\tau| , \tag{44}$$

where n is the greater of $1/N_2$ and $-1/M_3$.

Now for $\tau\neq 0$, and (f, x_2, x_3) in R, the partial derivatives $\partial f/\partial\tau$, $\partial x_2/\partial\tau$, and $\partial x_3/\partial\tau$ satisfy the equations of variation

$$\begin{aligned}\frac{\partial}{\partial t}\cdot\frac{\partial f}{\partial\tau} &= \frac{\partial f}{\partial\tau}(m_4+\omega T_{11}) + \frac{\partial x_2}{\partial\tau}\omega T_{12} + \frac{\partial x_3}{\partial\tau}\omega T_{13},\\ \frac{\partial}{\partial t}\cdot\frac{\partial x_2}{\partial\tau} &= \frac{\partial f}{\partial\tau}\omega T_{21} + \frac{\partial x_2}{\partial\tau}(m_2+\omega T_{22}) + \frac{\partial x_3}{\partial\tau}\omega T_{33},\\ \frac{\partial}{\partial t}\cdot\frac{\partial x_3}{\partial\tau} &= \frac{\partial f}{\partial\tau}\omega T_{31} + \frac{\partial x_2}{\partial\tau}\omega T_{32} + \frac{\partial x_3}{\partial\tau}(m_3+\omega T_{33}),\end{aligned} \tag{45}$$

where $\omega=x_2^{\lambda-1}x_3^{\lambda-1}$ and the T_{ij}, since every S_i has $x_2^\lambda x_3^\lambda$ as a factor, are defined by the equations

$$\begin{aligned}\omega T_{11} &= S_3+f\partial S_3/\partial f, & \omega T_{12} &= f\partial S_3/\partial x_2+\partial S_4/\partial x_2, & \omega T_{13} &= f\partial S_3/\partial x_3+\partial S_4/\partial x_3,\\ \omega T_{21} &= x_2x_3\partial S_1/\partial f, & \omega T_{22} &= x_3S_1+x_2x_3\partial S_1/\partial x_2, & \omega T_{23} &= \partial(x_2x_3S_1)/\partial x_3,\\ \omega T_{31} &= x_2x_3\partial S_2/\partial f, & \omega T_{32} &= \partial(x_3x_2S_2)/\partial x_2, & \omega T_{33} &= x_2S_2+x_3x_2\partial S_2/\partial x_3.\end{aligned}$$

Now for $t=0$, $\partial f/\partial\tau=0$, $\partial x_2/\partial\tau=\partial x_3/\partial\tau=1$. With this in mind let us set

$$\xi = \exp\left[\int_0^t (m_0+\omega T_{22})dt\right],\ \eta=\exp\left[\int_0^t (m_3+\omega T_{33})dt\right], \tag{46}$$
$$u=\partial f/\partial\tau,\quad v+\xi=\partial x_2/\partial\tau,\quad w+\eta=\partial x_3/\partial\tau.$$

Hence, when $t=0$, $y=v=w=0$. Now make use of the transformation (46) in the equations (45) and obtain

$$\begin{aligned}\frac{\partial u}{\partial t} &= u(m_4+\omega T_{11}) + (v+\xi)\omega T_{12} + (w+\eta)\omega T_{13},\\ \frac{\partial v}{\partial t} &= u\omega T_{21} + v(m_2+\omega T_{22}) + (w+\eta)\omega T_{23},\\ \frac{\partial w}{\partial t} &= u\omega T_{31} + (v+\xi)\omega T_{32} + w(m_3+\omega T_{33}).\end{aligned} \tag{47}$$

Hence

$$u = \exp\left[\int_0^t (m_4 + \omega T_{11})dt\right]\int_0^t \exp\left[-\int_0^t (m_4 + \omega T_{11})dt\right] \cdot [(v+\xi)\omega T_{12} + (w+\eta)\omega T_{13}]dt,$$

$$(48) \qquad v = \xi \int_0^t \xi^{-1}[u\omega T_{21} + (w+\eta)\omega T_{23}]dt,$$

$$w = \eta \int_0^t \eta^{-1}[u\omega T_{31} + (v+\xi)\omega T_{32}]dt.$$

We shall limit the discussion for the time being to that part of the first quadrant of the x_2x_3 plane which corresponds to $t \geqq 0$ since an argument similar to that about to be given can be made when $t \leqq 0$. Define $U = \max |u|$ for τ fixed and t variable, and make corresponding definitions for V and W. Also define the constants M_{ij} so that $\infty > M_{ij} > n \max |x_2^{\lambda-1}T_{ij}|$ in R, where n is that in (44). Since x_3 decreases when t increases, we have for this region that x_3 is less than its corresponding τ. From the first equation of (48) we obtain

$$\begin{aligned} U &\leqq e^{M_1 t}[\{VM_{12} + WM_{13}\}\tau^{\lambda-1} + \{e^{M_2 t}M_{12} + M_{13}\}\tau^{\lambda-1}]\,|\log\tau| \\ (49) \qquad &\leqq e^{nM_1|\log\tau|}[VM_{12} + WM_{13} + e^{nM_2|\log\tau|}M_{12} + M_{13}]\tau^{\lambda-1}|\log\tau| \\ &\leqq \tau^{-nM_1+\lambda-1}|\log\tau|\,[VM_{12} + WM_{13} + \tau^{-nM_2}M_{12} + M_{13}]. \end{aligned}$$

In a similar manner from the other two equations of (48) we obtain

$$(50) \qquad \begin{aligned} V &\leqq \tau^{-nM_2+\lambda-1}|\log\tau|\,[UM_{21} + WM_{23} + M_{23}], \\ W &\leqq \tau^{nN_3+\lambda-1}|\log\tau|\,[UM_{31} + VM_{32} + \tau^{-nM_2}M_{32}]. \end{aligned}$$

For $t \geqq 0$, if λ were to satisfy the inequality

$$(51) \qquad \lambda > 3 + n(M_1 + M_2 - N_3)$$

we could show that U, V and W approach zero as $\tau \to 0$ and hence that the partial derivatives $\partial x_2/\partial\tau$ and $\partial x_3/\partial\tau$ are greater than zero for τ less than a certain number which is greater than zero.

The inequality which corresponds to (51) for $t \leqq 0$ is the same as (51). Thus it follows from the definition of n and from (51) that it would be sufficient if λ were to satisfy the inequality

$$(52) \qquad \lambda > 3 + (M_1 + M_2 - N_3)(1/N_2 - 1/M_3).$$

But M_1, M_2 and N_3 depend to some extent on the choice of λ. Let us choose λ so large that

$$\lambda > 3 + L(m_4 + m_2 - m_3)(1/n_2 - 1/m_3)^*$$

* λ will also have to satisfy another inequality which arises from the determination of the other invariant surface.

where L is any positive number greater than 1, and then choose R so that (52) holds, which evidently can be done.

But, since $\tau<1$, the max $|\log\tau|\tau$ for the region R is when $\tau=e^{-1}$. Thus for $\tau<e^{-1}$ we may write the inequalities (49) and (50) as

$$
\begin{aligned}
U &\leqq \tau(VM_{12}+WM_{13})+\tau(M_{12}+M_{13}),\\
V &\leqq \tau(UM_{21}+WM_{23})+\tau M_{23},\\
W &\leqq \tau(UM_{31}+VM_{32})+\tau M_{32}.
\end{aligned}
\tag{53}
$$

These inequalities may be written in the form

$$
\begin{aligned}
U-\tau M_{12}V-\tau M_{13}W &\leqq \tau(M_{12}+M_{13}),\\
-\tau M_{21}U+V-\tau M_{23}W &\leqq \tau M_{23},\\
-\tau M_{31}U-\tau M_{32}V+W &\leqq \tau M_{32}.
\end{aligned}
$$

Since the determinant of the coefficients of U, V, W in the left hand members of these inequalities is definitely positive for τ small enough and since each of the right hand members contains τ as a factor and the M_{ij} are constants independent of τ, we see that for τ small enough U, V and W are as small as we desire.

This argument, with exception of that part of it relative to the choice of λ, has been made for $t\geqq 0$. A similar argument can be made for $t\leqq 0$ and hence as is seen from the equations (46), for τ small enough $\partial x_2/\partial\tau$ and $\partial x_3/\partial\tau$ are greater than zero for (f, x_2, x_3) in R and, as is still assumed, $x_2>0$ and $x_3>0$.

Now examine the jacobian for the last two equations of (43). In R and for $x_2>0$ and $x_3>0$ we have always $\partial x_2/\partial t$ greater than zero and $\partial x_3/\partial t$ less than zero. From these facts and those proved in the preceding paragraphs we see that we can find an open region R_0 defined by $x_2>0$, $x_3>0$, $x_2^2+x_3^2<2\tau_0^2$ where τ_0 is a constant greater than zero but so small that the jacobian

$$
\begin{vmatrix}
\partial x_2/\partial\tau & \partial x_3/\partial\tau\\
\partial x_2/\partial t & \partial x_3/\partial t
\end{vmatrix}
$$

is always less than zero. Hence for this region R_0 we can always solve the last two equations of (43) for t and τ as analytic functions of x_2 and x_3 and thus obtain $f=f(x_2, x_3)$ as has already been described.

We wish now to show that the limits of $f(x_2, x_3)$ and all its derivatives as $x_2\rightarrow 0$ or as $x_3\rightarrow 0$ are zero. To do this let us note that proving any function $G(x_2, x_3)$ satisfies the relationship

$$
G(x_2, x_3) = O(x_2^p x_3^q)
\tag{54}
$$

for an arbitrary pair of positive integers p, q is equivalent to proving that it satisfies the relationship

$$G(x_2, x_3) = O(\tau^r) \tag{55}$$

for an arbitrary positive integer r. It is understood that x_2 and x_3 are positive and that τ is the value of the parameter which corresponds to the point (x_2, x_3) by means of the last two equations of (43).

First of all let us observe that the last two equations of (40) imply that

$$\tau/x_2 \leqq \tau^{-nM_2}, \qquad \tau/x_3 \leqq \tau^{nN_3}. \tag{56}$$

Now consider that part of the quadrant in the x_2x_3 plane which corresponds to $t \geqq 0$. For this region $x_3 \leqq \tau$, and the equation (54) implies the existence of a positive constant M_q such that $|G(x_2, x_3)| \leqq x_3{}^q M_q$, where q is any positive integer. Hence $|G(x_2, x_3)| \leqq \tau^q M_q$. In the same way we can show that for that part of the quadrant which corresponds to $t \leqq 0$ there exists a positive constant M_p corresponding to any arbitrary positive integer p such that $|G(x_2, x_3)| \leqq \tau^p M_p$. If we define M_{pq} as the greater of M_p and M_q we see that for t positive or negative

$$|G(x_2, x_3)| \leqq \tau^r M_{rr} \tag{57}$$

for an arbitrary positive integer r, the M_{rr} being a positive constant depending on the r. Hence the relationship (54) implies the relationship (55).

Now consider the converse situation. The equation (55) implies the existence of a positive constant M_{rr} and a relationship of the form (57) for an arbitrary positive integer r. On using the inequalities (56) we obtain the inequalities

$$\frac{|G|}{x_2{}^p x_3{}^q} \leqq \frac{\tau^p}{x_2{}^p} \cdot \frac{\tau^q}{x_3{}^q} M_{rr} \tau^{r-p-q} \leqq M_{rr} \tau^{r-p-q-pnM_2+qnN_3}.$$

If we choose r so large that $r-p-q-pnM_2+qnN_3>0$, the quantity after the last inequality sign is finite and hence we have the result that the relationship (55) implies the relationship (54).

We are now in a position to discuss the behavior of $f(x_2, x_3)$ and of its partial derivatives as $x_2 \to 0$ or as $x_3 \to 0$. From the first equation of (40) and from (39) and (42) we have that

$$f = \exp\left[\int_0^t (m_4 + S_3)\,dt\right] \int_0^t \exp\left[-\int_0^t (m_4 + S_3)\,dt\right] O(\tau^r)\,dt,$$

whence

$$|f| \leqq \tau^{-nM_1} |O(\tau^r)| \, n \, |\log \tau| \leqq |O(\tau^{r-nM_2-1})|$$

for r an arbitrary positive integer. Hence $f(x_2, x_3) = O(x_2{}^p x_3{}^q)$ for arbitrary positive integers p,q.

In order to discuss the behavior of $\partial f/\partial x_2$ as $x_2\to 0$ or as $x_3\to 0$ we shall make use of the fact that

$$\frac{\partial f}{\partial x_2}=\frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial x_2}+\frac{\partial f}{\partial \tau}\cdot\frac{\partial \tau}{\partial x_2}\cdot \tag{58}$$

Since $f(x_2, x_3)=O(x_2{}^p x_3{}^q)$ for arbitrary p, q as has just been proved, the first equation of (40) shows that $\partial f/\partial t$ has the same property. Now consider the equations (45) once again. From them we have proved that $\partial x_2/\partial \tau$ behaves like ξ of (46) for τ small enough and hence for x_2 or x_3 small enough. Hence $\partial x_2/\partial\tau$, as $x_2\to 0$ or as $x_3\to 0$, is at most infinite like τ^{-nM_1}. Similarly we have $\partial x_3/\partial\tau=O(\tau^{nN_1})$. Now we are in a position to examine the behavior of $\partial\tau/\partial x_2$ and of $\partial t/\partial x_2$. To do this let us consider the identities

$$\begin{aligned}1&=\frac{\partial\tau}{\partial x_2}\cdot\frac{\partial x_2}{\partial\tau}+\frac{\partial\tau}{\partial x_3}\cdot\frac{\partial x_3}{\partial\tau},\\ 0&=\frac{\partial\tau}{\partial x_2}\cdot\frac{\partial x_2}{\partial t}+\frac{\partial\tau}{\partial x_3}\cdot\frac{\partial x_3}{\partial t},\end{aligned}\tag{59}$$

and

$$\begin{aligned}1&=\frac{\partial t}{\partial x_2}\cdot\frac{\partial x_2}{\partial t}+\frac{\partial t}{\partial x_3}\cdot\frac{\partial x_3}{\partial t},\\ 0&=\frac{\partial t}{\partial x_2}\cdot\frac{\partial x_2}{\partial\tau}+\frac{\partial t}{\partial x_3}\cdot\frac{\partial x_3}{\partial\tau}\cdot\end{aligned}\tag{60}$$

Since the determinant

$$\begin{vmatrix}\partial x_2/\partial\tau & \partial x_3/\partial\tau\\ \partial x_2/\partial t & \partial x_3/\partial t\end{vmatrix}$$

is zero only when $x_2=x_3=0$ and then vanishes like τ, we see that $\partial t/\partial x_2$ and $\partial\tau/\partial x_2$ are at most infinite like a finite power of τ^{-1} as $x_2\to 0$ or as $x_3\to 0$. Let this power be m so that we have that $\partial t/\partial x_2$ and $\partial\tau/\partial x_2$ are $O(\tau^{-m})$ where m is some finite positive number.

Now return to the first differential equation of (45) and note that in addition to the facts used in the discussion there made T_{12} and T_{13} are $O(x_2{}^p x_3{}^q)$ for arbitrary positive integers p, q. On integrating this equation and using the inequalities just established we obtain the result that

$$\left|\frac{\partial f}{\partial\tau}\right|=\left|\exp\left[\int_0^t(m_4+\omega T_{11})dt\right]\int_0^t\exp\left[-\int_0^t(m_4+\omega T_{11})dt\right]\right.$$
$$\left.\cdot\left[\frac{\partial x_2}{\partial\tau}\omega T_{12}+\frac{\partial x_3}{\partial\tau}\omega T_{13}\right]dt\right|\leqq\tau^{-nM_1}\left|O(\tau^p)\right|n\left|\log\tau\right|=\left|O(\tau^r)\right|$$

for an arbitrary positive integer r.

Out of this discussion and the equation (58) follows immediately the fact that $\partial f/\partial x_2 = O(r^r) = O(x_2^p x_3^q)$ where r, p and q are arbitrary positive integers. It is evident by induction, using the equations (40), (59) and (60), that corresponding statements are true for all the partial derivatives of $f(x_2, x_3)$ with respect to x_2 and x_3 and hence that all these partial derivatives are zero for $x_2 = 0$ or for $x_3 = 0$.

Although the function $f(x_2, x_3)$ which we have just been discussing has been defined only for the quadrant of R for which x_2 and x_3 are positive, it is readily seen that it can be defined in the other quadrants in a similar manner and will have the same properties with respect to these quadrants as it has with respect to the first one. When these four functions are joined along the axes they unite to form a function which is of class C^∞ in a neighborhood of the origin, is analytic for $x_2 \neq 0$, $x_3 \neq 0$, and is zero together with all its partial derivatives when $x_2 = 0$ or when $x_3 = 0$. This completes the proof of Lemma 3.

V. First non-analytic normalization

Thus we have found two invariant surfaces other than $x_3 = 0$. For purposes of notation let the equations of these invariant surfaces be

$$(61) \qquad \begin{aligned} x_1 &= x_2^2 x_3^2 f_1(x_2, x_3) &= F(x_2, x_3), \\ x_2 &= x_1^2 x_3^2 g_1(x_1, x_3) &= G(x_1, x_3). \end{aligned}$$

Reviewing the properties of F and G, we know that the function $F(x_2, x_3)$ is of class C^∞ in a neighborhood $|x_2|, |x_3| < r_f, r_f > 0$ of $x_2 = x_3 = 0$ and is analytic at any point (x_2, x_3) of this neighborhood for which $x_2 \neq 0$, $x_3 \neq 0$; the function $G(x_1, x_3)$ is of class C^∞ in a certain neighborhood, $|x_1|, |x_3| < r_g, r_g > 0$, of $x_1 = x_3 = 0$ and is analytic at a point (x_1, x_3) of this neighborhood for which $x_1 \neq 0$, $x_3 \neq 0$. Let r_0 be the smaller of r_f and r_g.

The aim of this section is to use these invariant surfaces in a transformation which will transform the differential equations (30) into ones of the same form in which the $Q_1 = Q_2 = 0$. Now the invariant surfaces (61) fail to be analytic along the axes and we wish to set up the equations of a transformation which will carry these surfaces into coördinate planes, and will be analytic except possibly for those points which lie on these surfaces. To this end let us examine the transformation

$$(62) \qquad \begin{aligned} x_1 &= y_1 + F[x_2 - G(y_1, y_3), y_3], \\ x_2 &= y_2 + G[x_1 - F(y_2, y_3), y_3], \\ x_3 &= y_3, \end{aligned}$$

and confine our attention in the x_i space to a region R_x about the origin for which $|x_i| < r_0$, and in the y_i space to a region R_y about the origin for which $|y_i| < r_0$. Furthermore, let R_x and R_y be so chosen that the transformation (62) establishes a one-to-one correspondence between the points of R_x and those of R_y, which clearly can be done so that the origin $(x_1, x_2, x_3) = (0, 0, 0)$ is interior to R_x and the origin $(y_1, y_2, y_3) = (0, 0, 0)$, which happens to correspond to the origin in the x_i space, is interior to R_y. From the properties of $F(x_2, x_3)$ and of $G(x_1, x_3)$ it follows that when $y_1 = 0$, $x_1 = F(x_2, x_3)$, and when $y_2 = 0$, $x_2 = G(x_1, x_3)$. Moreover, this transformation (62) is of class C^∞ and can fail to be analytic only when $y_1 = 0$, $y_2 = 0$, or $y_3 = 0$.

On solving the equations (62) for x_i we obtain

$$x_1 = f_1(y_1, y_2, y_3), \quad x_2 = f_2(y_1, y_2, y_3), \quad x_3 = y_3, \tag{63}$$

where the functions f_i are of class C^∞ in R_y and are analytic at any point of R_y for which $y_1 \neq 0$, $y_2 \neq 0$, $y_3 \neq 0$. Furthermore, in any one of the surfaces $y_1 = 0$, $y_2 = 0$, or $y_3 = 0$, the functions f_i are of class C^∞ in the variables of the surface, analytic at any point which does not lie on either of the axes in the surface, and are analytic along either axis in the variable of the axis in a neighborhood of the origin. This statement is obvious for $y_3 = 0$. For $y_1 = 0$, $x_1 = F(y_2, y_3)$ and $x_2 = y_2$. On account of the properties of the $F(x_2, x_3)$ the statement is seen to be true when $y_1 = 0$. In a similar manner it can be shown to be true when $y_2 = 0$.

Since the f_i are of class C^∞ in the R_y we may formally expand these functions as power series in y_3 or as power series in y_1 and y_2 together. Let these expansions be written as

$$\begin{aligned} x_i &\sim \alpha_{i0}(y_1, y_2) + \alpha_{i1}(y_1, y_2) y_3 + \cdots \\ &= \alpha_{i00}(y_3) + \alpha_{i10}(y_3) y_1 + \alpha_{i01}(y_3) y_2 + \cdots, \end{aligned} \tag{64}$$

$i = 1, 2$. We shall now prove

LEMMA 4. *The $\alpha_{ij}(y_1, y_2)$ of* (64) *are analytic in y_1, y_2 for $|y_1|$, $|y_2| < r_0$, and the $\alpha_{ijk}(y_3)$ are analytic in y_3 for $|y_3| < r_0$.*

Since $F(x_2, x_3)$ and $G(x_1, x_3)$ each contains x_3^2 as a factor, it is seen that $\alpha_{i0} = y_i$, $i = 1, 2$. We shall denote the partial derivative of F with respect to its first argument by F_1, and with respect to its second by F_2. Furthermore, when the arguments of F are y_2, y_3 we shall write it merely as F, but when the arguments are $x_2 - G(y_1, y_3)$, y_3 we shall write it as F^*. Corresponding notations will be adopted for G and its partial derivatives.

Now differentiate the first two equations of (62) partially with respect to y_3. Hence

$$\frac{\partial x_1}{\partial y_3} = F_2^* + F_1^*\left(\frac{\partial x_2}{\partial y_3} - G_2\right), \quad \frac{\partial x_2}{\partial y_3} = G_2^* + G_1^*\left(\frac{\partial x_1}{\partial y_3} - F_2\right),$$

whence

$$\begin{aligned} &\frac{\partial x_1}{\partial y_3}(1 - F_1^*G_1^*) = F_2^* + F_1^*G_2^* - F_1^*G_1^*F_2 - F_1^*G_2, \\ (65)\qquad &\frac{\partial x_2}{\partial y_3}(1 - F_1^*G_1^*) = G_2^* + G_1^*F_2^* - G_1^*F_1^*G_2 - G_1^*F_2. \end{aligned}$$

Now for $y_3=0$, F_1G_1 and the right hand sides of the equations (65) are zero since F and G both contain x_3^2 as a factor. Hence $\partial x_1/\partial y_3$ and $\partial x_2/\partial y_3$ are both zero for $y_3=0$.

Now recall the fact that F, as well as all its partial derivatives, is analytic in x_2 for $|x_2|<r_0$, $x_3=0$, and that G and its partial derivatives have corresponding properties. Hence any partial derivative of any order of F^* or G^* with respect to the arguments x_2-G, y_3, and x_1-F, y_3, respectively, is analytic in y_2 and y_1, respectively, for $|y_i|<r_0$, $y_3=0$, since $y_1=x_1$ and $y_2=x_2$ for $y_3=0$. The nth order partial derivatives of F_i^* and G_i^* with respect to y_3 are polynomials in the partial derivatives of x_1 and x_2 with respect to y_3 up to and including those of the nth order. The coefficients in these polynomials are polynomials in F_j^*, G_j^*, and their partial derivatives with respect to their respective arguments. Hence, from the equations (65) we may determine seriatim all the partial derivatives of x_1 and x_2 with respect to y_3 at $y_3=0$ as functions of y_1, y_2 which are analytic for $|y_1|$, $|y_2|<r_0$.

Now differentiate the first two equations of (62) partially with respect to y_1 and y_2. We obtain on simplification

$$\begin{aligned} &\frac{\partial x_1}{\partial y_1}[1 - F_1^*G_1^*] = 1 - F_1^*G_1, \qquad \frac{\partial x_2}{\partial y_1} = G_1^*\frac{\partial x_1}{\partial y_1}, \\ (66)\qquad &\frac{\partial x_2}{\partial y_2}[1 - F_1^*G_1^*] = 1 - G_1^*F_1, \qquad \frac{\partial x_1}{\partial y_2} = F_1^*\frac{\partial x_2}{\partial y_2}. \end{aligned}$$

Now when $y_1=y_2=0$, $x_1=x_2=0$. Hence, by using an argument similar to that used in connection with the partial derivatives with respect to y_3, it is seen that from (66) we may determine seriatim all the partial derivatives of x_1 and of x_2 with respect to y_1 and y_2 at $y_1=y_2=0$ as functions of y_3 which are analytic for $|y_3|<r_0$. This completes the proof of the lemma.

From the equations (65) and (66) we can now prove that all the partial derivatives of x_1 and x_2, when y_1, y_2, or y_3 is zero, are of class C^∞ in the remaining variables and are analytic in these variables provided neither is zero. We shall prove this only for the $\partial x_1/\partial y_3$ for $y_1=0$ because an induction proof can be made for all the other derivatives except those of the first order and a proof can be given for them along the lines of that to follow. As we have already seen, if $y_1=0$, then $x_2=y_2$, $x_3=y_3$, and $x_1=F(y_2, y_3)$. Hence the coefficient of $\partial x_1/\partial y_3$ in the first equation of (65) is 1 for $y_1=0$, and it is readily verified that the right hand member of this equation is of class C^∞ in y_2 and y_3, and is analytic in these variables for $y_2\neq 0$, $y_3\neq 0$.

Now let us perform the transformation of variables defined by the equations (62) or the equations (63). The differential equations (30) will be transformed into differential equations of the same form in y_i since under the transformation (62) the $x_1=x_2=0$ axis and the $x_3=0$ plane are invariant. From the definition of $y_1=0$ it follows that when an integral curve has at any point $y_1=0$, it has $y_1=0$ for all its points in the neighborhood of the origin under consideration. A similar statement is true concerning y_2. Hence the right hand members of the first two equations in our new system have y_1 and y_2, respectively, as factors. Thus the new equations have the form

$$(67)\qquad \begin{aligned} \frac{dy_1}{dt} &= y_1(m_1 + y_3U_1),\\ \frac{dy_2}{dt} &= y_2(m_2 + y_3U_2),\\ \frac{dy_3}{dt} &= y_3(m_3 + y_1U_3 + y_2U_4), \end{aligned}$$

in which the U_i are functions of the y_j which are of class C^∞ in a neighborhood of the origin and are analytic at any point of this neighborhood for which $y_j\neq 0, j=1, 2, 3$. Furthermore, the U_i and all their partial derivatives have the property that in any one of the coördinate planes $y_j=0, j=1, 2, 3$, they are of class C^∞ in a neighborhood of the origin, analytic at any point of this neighborhood which does not lie on the coördinate axes of this plane, and analytic along any one of these coördinate axes in the variable of the axis in a neighborhood of the origin. All these statements follow at once from the properties of the transformation (63), because on differentiating the equations of (63) we obtain

$$(68)\qquad \begin{aligned} \frac{dx_1}{dt} &= \frac{\partial f_1}{\partial y_1}\cdot\frac{dy_1}{dt} + \frac{\partial f_1}{\partial y_2}\cdot\frac{dy_2}{dt} + \frac{\partial f_1}{\partial y_3}\cdot\frac{dy_3}{dt} = x_1(m_1 + x_3P_1) + x_2^2x_3^2Q_1,\\ \frac{dx_2}{dt} &= \frac{\partial f_2}{\partial y_1}\cdot\frac{dy_1}{dt} + \frac{\partial f_2}{\partial y_2}\cdot\frac{dy_2}{dt} + \frac{\partial f_2}{\partial y_3}\cdot\frac{dy_3}{dt} = x_2(m_2 + x_3P_2) + x_1^2x_3^2Q_2,\\ \frac{dx_3}{dt} &= \frac{\partial f_3}{\partial y_1}\cdot\frac{dy_1}{dt} + \frac{\partial f_3}{\partial y_2}\cdot\frac{dy_2}{dt} + \frac{\partial f_3}{\partial y_3}\cdot\frac{dy_3}{dt} = x_3(m_3 + x_1P_3 + x_2P_4). \end{aligned}$$

Now replace the x_i in these equations by the functions of y_j as given in (63). Since all the functions of y_j which now appear in these equations (68) as well as all their partial derivatives have the same properties of continuity and of analyticity that the f_i themselves have, we see at once that our statement concerning the U_i and their partial derivatives is true.

VI. Second non-analytic normalization

The aim of this section is to reduce the system of differential equations (67) to an equivalent system of the same form for which the formal series for the ϕ_i will consist in the linear terms only. To this end we shall first prove

Lemma 5. *Consider the formal series*

$$(69)\qquad \begin{aligned} \psi &\sim x_1 + \cdots + \frac{a_{mnp}}{m!n!p!} x_1^m x_2^n x_3^p + \cdots \\ &= x_1 + \alpha_1(x_1, x_2)x_3 + \cdots + \alpha_p(x_1, x_2)x_3^p + \cdots \\ &= x_1 + \cdots + \alpha_{mn}(x_3)x_1^m x_2^n + \cdots \end{aligned}$$

in which the only linear term is x_1, every other term contains x_1x_3 or x_2x_3 as a factor, α_p are analytic in x_1, x_2 for $|x_1|$, $|x_2|<r$, $r>0$, and α_{mn} are analytic in x_3 for $|x_3|<r$. Into the series (69) *substitute the formal series*

$$(70)\qquad \begin{aligned} x_i &\sim y_i + \beta_{i1}(y_1, y_2)x_3 + \cdots + \beta_{ip}(y_1, y_2)x_3^p + \cdots \\ &= y_i + \cdots + \beta_{imn}(x_3)y_1^m y_2^n + \cdots, \quad i = 1, 2, \end{aligned}$$

in which the only linear term in the ith series is y_i, every other term contains y_1x_3 or y_2x_3 as a factor, the β_{ip} are analytic in y_1, y_2 for $|y_1|$, $|y_2|<r$ and β_{imn} are analytic in x_3 for $|x_3|<r$, and arrange the resulting series as

$$(71)\qquad \begin{aligned} \psi &\sim \gamma_0(y_1, y_2) + \cdots + \gamma_p(y_1, y_2)x_3^p + \cdots \\ &= \gamma_{00}(x_3) + \cdots + \gamma_{mn}(x_3)y_1^m y_2^n + \cdots. \end{aligned}$$

Then γ_i are analytic in y_1, y_2 for $|y_1|$, $|y_2|<r$ and γ_{mn} are analytic in x_3 for $|x_3|<r$.

The truth of this lemma is almost obvious. From the properties of the series (69) and (70) it follows at once that $\gamma_0 = y_1$ and that $\gamma_{00}=0$. It readily follows that $\gamma_1 = \beta_{11}(y_1, y_2) + \alpha_1(y_1, y_2)$ which is obviously analytic for $|y_1|$, $|y_2|<r$. Seriatim, we may prove that all the γ_p are analytic for $|y_1|$, $|y_2|<r$, and in a similar manner we may prove that all the γ_{mn} are analytic for $|x_3|<r$.

Now consider the effect of the transformation (62) on the formal series

ϕ_i. From Lemmas 4 and 5 it follows that the series ϕ_i when expanded in powers of y_j have the property that when they are arranged as power series in y_3 the coefficients are analytic in y_1 and y_2 for $|y_1|$, $|y_2|<r_0$, and when arranged as power series in y_1 and y_2 the coefficients are analytic in y_3 for $|y_3|<r_0$. But these series in y_j obtained by substituting the series of (64) for the x_j in the series for the ϕ_i are the same as the formal series solutions of the partial differential equations

$$\begin{aligned} \frac{\partial\phi_i}{\partial y_1}y_1(m_1+y_3U_1)&+\frac{\partial\phi_i}{\partial y_2}y_2(m_2+y_3U_2)\\ &+\frac{\partial\phi_i}{\partial y_3}y_3(m_3+y_1U_3+y_2U_4)=m_i\phi_i \end{aligned} \tag{72}$$

since the equations (72) are the transforms of the equations (31) under the transformation (62). From the form of the equations (72) it follows that the formal series for ϕ_i has y_i as a factor for $i=1, 2, 3$, respectively. Hence we may write

$$\phi_i \sim y_i\left(1+\cdots+\frac{a_{imnp}}{m!n!p!}y_1^m y_2^n y_3^p+\cdots\right) \qquad (i=1,2,3), \tag{73}$$

in which the series within the brackets have the same properties relative to the convergence of sets of terms as the series for the ϕ_i themselves have. Furthermore, the series for ϕ_1 and ϕ_2 reduce to y_1 and y_2, respectively, when $y_3=0$, and the series for ϕ_3 reduces to y_3 when $y_1=y_2=0$. This follows immediately from the form of the partial differential equations (72).

Now "fit" the series within the parentheses appearing in the equations (73) by means of three functions $F_i(y_1, y_2, y_3)$ in the manner described in Lemma 2, leaving the constant terms as they are. Hence we have the transformation

$$z_i = y_i(1+F_i(y_1,y_2,y_3)) \qquad (i=1,2,3), \tag{74}$$

defined in which the F_i have the properties relative to the ϕ_i which are peculiar to their mode of definition.

We wish now to determine the type of the system of differential equations which the z_i satisfy when the y_j in the equations (74) are replaced by an arbitrary solution $y_j(t)$ of the differential equations (67). On account of the properties of the power series ϕ_i which were mentioned in a preceding paragraph, we see that the transformation (74) has the property that if $y_1=y_2=0$, then $z_1=z_2=0$ and $z_3=y_3$, and if $y_3=0$, then $z_3=0$, $z_1=y_1$ and $z_2=y_2$. Hence the form of the differential equations which the z_i satisfy is the same as the form of the differential equations which the y_i satisfy, since also $y_i=0$ implies that $z_i=0$ for $i=1, 2, 3$.

On remembering that

$$\frac{dz_i}{dt} = \frac{\partial z_i}{\partial y_1} y_1(m_1 + y_3 U_1) + \frac{\partial z_i}{\partial y_2} y_2(m_2 + y_3 U_2) + \frac{\partial z_i}{\partial y_3} y_3(m_3 + y_1 U_3 + y_2 U_4),$$

and on recalling the definition of the z_j we see that

$$\frac{dz_i}{dt} - m_i z_i = R_i \qquad (i = 1,2,3),$$

where the R_i are functions of the y_j which have the same properties as have the F_i in a neighborhood of the origin and are zero together with all their partial derivatives for $y_1 = y_2 = 0$ or for $y_3 = 0$. From these properties of the R_i and those of the transformation (74) it follows that the functions R_i and all their partial derivatives with respect to the z_j are zero when $z_1 = z_2 = 0$ or when $z_3 = 0$.

From this argument follows

LEMMA 6. *Under the transformation* (74) *the system of differential equations* (67) *is equivalent to a system of the type*

$$\frac{dz_1}{dt} = z_1(m_1 + z_3 T_1),$$
$$\frac{dz_2}{dt} = z_2(m_2 + z_3 T_2), \tag{75}$$
$$\frac{dz_3}{dt} = z_3(m_3 + z_1 T_3 + z_2 T_4),$$

where T_i are functions of z_j, are of Class C^∞ in a neighborhood of $z_1 = z_2 = z_3 = 0$, are analytic for $z_1 \neq 0$, $z_2 \neq 0$, $z_3 \neq 0$, are analytic together with all their partial derivatives in z_i, z_j for $z_k = 0$, $i \neq j$, $i \neq k$, $j \neq k$, and are zero together with all their partial derivatives when $z_1 = z_2 = 0$ or when $z_3 = 0$.

It may be noted here and will be found of use in a later section that the properties of the T_i imply that each is $O[(z_1^2 + z_2^2)^p z_3^q]$ for any positive integers p, q.

VII. Final reduction

In this section we shall make the final reduction, i.e., we shall find a transformation which will reduce the equations (75) to the form (8). The finding of such a transformation will be effected when a certain set of three functions is determined. Since all three of these functions can be determined in exactly the same way, we shall limit ourselves to the determination of the first.

In fact we shall prove the following lemma by showing only how to obtain the first of the functions g_i:

LEMMA 7. *There exists a transformation of the type*

$$z_i^* = z_i g_i(z_1, z_2, z_3) \qquad (i = 1,2,3),$$

where the functions $g_i(z_1, z_2, z_3)$ *are of class* C^∞ *in a neighborhood of the origin and are analytic for* $z_1 \neq 0$, $z_2 \neq 0$, $z_3 \neq 0$, *under which the differential equations* (75) *are equivalent to the differential equations*

$$\frac{dz_i^*}{dt} = m_i z_i^* \qquad (i = 1,2,3).$$

Furthermore, under this transformation the plane $z_i = 0$ *corresponds to the plane* $z_i^* = 0$ *for* $i = 1, 2, 3$.

The discussion will be relative to a closed region S of the z_i space which contains the origin as an interior point and in which $z_1^2 + z_2^2 < 1, z_3^2 < 1, m_1 + z_3 T_1$ and $m_2 + z_3 T_2$ are greater than zero and $m_3 + z_1 T_3 + z_2 T_4$ is less than zero. It will be further assumed that S is composed only of points which lie on integral curves of the differential equations of the system (75) which intersect the cone

$$(76) \qquad z_1^2 + z_2^2 - z_3^2 = 0.$$

It is readily seen that this hypothesis concerning S is compatible with the others, since, according to the others, the slopes dz_1/dz_3 and dz_2/dz_3 of any integral curve in S are zero or infinite only for those integral curves lying in the $z_3 = 0$ plane or in the $z_1 = z_2 = 0$ axis. This implies that the integral curves of the differential equations (75) passing through an arbitrary point in a sufficiently small neighborhood of the origin will intersect the cone (76) in S. On account of the signs of the left hand members of the equations (75) in S, it is seen that every integral curve in S with the exception of those lying in the $z_3 = 0$ plane or in the $z_1 = z_2 = 0$ axis intersects the cone in only one point. It will be found convenient to define S_0 as S with the plane $z_3 = 0$ and the axis $z_1 = z_2 = 0$ removed.

If we write the equations of the cone (76) in the parametric form

$$(77) \qquad z_1 = \tau_1, \quad z_2 = \tau_2, \quad z_3 = \pm (\tau_1^2 + \tau_2^2)^{1/2},$$

and choose these equations as the equations giving the initial values of the z_i when $t = 0$, we see that the points in S_0 are placed in a one-to-one correspondence with the points in the corresponding τ_1, τ_2, t space by means of the equations of the integral curves of the differential equations (75),

$$(78) \qquad z_i = z_i(\tau_1, \tau_2, t) \qquad (i = 1,2,3),$$

where the functions z_i are of class C^∞ in their arguments for (z_1, z_2, z_3) in S_0, and are analytic if $\tau_1 \neq 0$, $\tau_2 \neq 0$.

Let us introduce the following notation relative to the region S:

$$M_1 = \max(m_1 + z_3T_1),\ M_2 = \max(m_2 + z_3T_2),\ M_3 = \max(m_3 + z_1T_3 + z_2T_4),$$
(79)
$$N_1 = \min(m_1 + z_3T_1),\ N_2 = \min(m_2 + z_3T_2),\ N_3 = \min(m_3 + z_1T_3 + z_2T_4).$$

Let M_0 be the larger of M_1 and M_2, and let N_0 be the smaller of N_1 and N_2.

For this region we wish to find how the t of any point of S_0 varies with the $\tau_1^2 + \tau_2^2$ of this point. From the first two equations of (75) it follows that

$$\frac{d(z_1^2 + z_2^2)}{z_1^2 + z_2^2} = \frac{2z_1^2(m_1 + z_3T_1) + 2z_2^2(m_2 + z_3T_2)}{z_1^2 + z_2^2}dt.$$

Hence

$$\log(z_1^2 + z_2^2) - \log(\tau_1^2 + \tau_2^2) = \int_0^t \frac{2z_1^2(m_1 + z_3T_1) + 2z_2^2(m_3 + z_3T_2)}{z_1^2 + z_2^2}dt.$$

Since $z_1^2 + z_2^2$ is less than 1 in S, $\log(z_1^2 + z_2^2)$ is negative. Consider that portion of S_0 which corresponds to $t \geqq 0$. For this portion $z_1^2 + z_2^2$ is greater than $\tau_1^2 + \tau_2^2$. Hence $t \leqq -[\log(\tau_1^2 + \tau_2^2)]\ [1/(2N_0)]$ since N_0 is less than the integrand of the integral in the above equation.

By means of a similar argument with respect to the third equation of (75) for $t \leqq 0$ we can show that $t \geqq -[\log(\tau_1^2 + \tau_2^2)]\ [1/(2N_3)]$. If we let N be the smaller of $2N_0$ and $|2N_3|$ we see that

(80) $$|t| \leqq -[\log(\tau_1^2 + \tau_2^2)][1/N],$$

where (τ_1, τ_2, t) corresponds to the point (z_1, z_2, z_3) in S_0.

Now the first equation of (75) is transformed into one of the same form by any transformation of the type

(81) $$z_1^* = z_1 g_1(z_1, z_2, z_3), \quad z_2^* = z_2, \quad z_3^* = z_3,$$

where $g_1(0, 0, 0)$ is 1. This follows from the fact that this transformation leaves the $z_1 = 0$ plane invariant. Thus we wish to determine the function $g_1(z_1, z_2, z_3)$ so that the corresponding z_1^* will satisfy the equation

(82) $$dz_1^*/dt = m_1 z_1^*,$$

when the z_i in the equations (81) are replaced by an arbitrary solution $z_i(t)$ of (75).

On differentiating the first equation of (81) with respect to t and using the fact that we wish z_1^* to satisfy the equation (82), we obtain

$$\frac{dz_1^*}{dt} = \frac{dz_1}{dt}g_1 + z_1\frac{dg_1}{dt} = z_1 g_1 m_1,$$

whence

$$\frac{d \log g_1}{dt} = - z_3 T_1.$$

If we set $f_1 = \log g_1$, it is evident that, if we can find a function $f_1 = f_1(z_1, z_2, z_3)$ of class C^∞ such that

$$\frac{df_1}{dt} = - z_3 T_1 \tag{83}$$

along every integral curve of (75) and such that $f_1(0, 0, 0)$ is zero, we have solved our problem because in the same way in which we found f_1 we can find functions f_2 and f_3 corresponding to z_2 and z_3, respectively, in the same way that f_1 corresponds to z_1.

Now determine the solution

$$f_1 = h_1(\tau_1, \tau_2, t) \tag{84}$$

of the differential equation (83) which corresponds to the solution (78) of (75) and which has the initial value $f_1 = 0$ for $t = 0$. On account of the fact that the right hand members of the differential equations (75) and (83) are of class C^∞ in their arguments and are analytic if $z_i \neq 0$, $i = 1, 2, 3$, the function h_1 of (84) is evidently of class C^∞ in its arguments for $\tau_1^2 + \tau_2^2 \neq 0$ and analytic for $\tau_1 \neq 0$, $\tau_2 \neq 0$. Since by means of the equations (78) we can express τ_1, τ_2 and t as single-valued functions of z_i of class C^∞ for (z_1, z_2, z_3) in S_0 which will be analytic if $z_i \neq 0$, $i = 1, 2, 3$, we see that we may write the equation (84) as $f_1 = f_1(z_1, z_2, z_3)$ where the function $f_1(z_1, z_2, z_3)$ will be defined only for (z_1, z_2, z_3) in S_0, and will be of class C^∞ in its arguments for this region, analytic if $z_i \neq 0$, $i = 1, 2, 3$. We need now only to investigate how this function behaves as $z_3 \to 0$ or as $(z_1, z_2) \to (0, 0)$. We shall prove that it and all its partial derivatives have limits zero as $z_3 \to 0$ or as $(z_1, z_2) \to (0, 0)$ and hence can be so defined that they are of class C^∞ for (z_1, z_2, z_3) in a certain neighborhood of $(0, 0, 0)$.

Let us first observe that, on account of the similarity of the situation here to that in Section IV, it can be readily shown that proving a function $f(z_1, z_2, z_3)$ to be $O[(z_1^2 + z_2^2)^p z_3^{2q}]$ is equivalent to proving it $O[(\tau_1^2 + \tau_2^2)^r]$ where p, q and r are arbitrary positive integers.

Now we are in a position to discuss the behavior of $f_1(z_1, z_2, z_3)$. It follows from the equation (83) and from the inequality (80) that

$$|f_1| = \left|\int_0^t O[(z_1^2 + z_2^2)^p z_3^{2q}]dt\right| \leqq |O[(\tau_1^2 + \tau_2^2)^r]| \cdot |\log(\tau_1^2 + \tau_2^2)| \cdot 1/N$$

for an arbitrary positive integer r, since $T_1 = O[(z_1^2 + z_2^2)^p z_3^{2q}]$ for arbitrary positive integers p, q. Hence $f_1(z_1, z_2, z_3) = O[(z_1^2 + z_2^2)^p z_3^{2q}]$ for arbitrary positive integers p, q.

By the method used in Section IV for the function $f(x_2, x_3)$ it can be shown that every partial derivative of $f_1(z_1, z_2, z_3)$ is also $O[(z_1^2 + z_2^2)^p z_3^{2q}]$ for arbitrary positive integers p, q, since all the partial derivatives $\partial z_1/\partial \tau_1$, $\partial \tau_1/\partial z_1$, $\partial z_1/\partial t$, etc., are at most infinite like a finite power of $(\tau_1^2 + \tau_2^2)^{-1}$. Hence the function $f_1(z_1, z_2, z_3)$ is of class C^∞ in a neighborhood of the origin and therefore, on account of its definition, the function $g_1(z_1, z_2, z_3)$ of (81) has the same property. Since the analytic properties of the f_1 also imply that the g_1 has corresponding analytic properties, and since the other functions g_2 and g_3 can be determined in the same way in which the g_1 has been determined and will have corresponding properties of continuity and of analyticity, we see that the proof of Lemma 7 is complete.

We may summarize the discussion of this and the preceding sections in the concluding

THEOREM 2. *There exists a transformation of the type*

$$x_i = x_i(z_1^*, z_2^*, z_3^*) \qquad (i = 1,2,3), \tag{85}$$

where the functions $x_i(z_1^*, z_2^*, z_3^*)$ *are of class* C^∞ *in some neighborhood* $(z_1^*, z_2^*, z_3^*) = (0, 0, 0)$ *and are analytic for* $z_1^* \neq 0$, $z_2^* \neq 0$, $z_3^* \neq 0$, *under which the differential equations* (7) *are equivalent to the differential equations*

$$\frac{dz_i^*}{dt} = m_i z_i^* \qquad (i = 1,2,3),$$

in a certain neighborhood of the origin, which point remains invariant under the transformation. Furthermore, when the equations (85) *are solved for the* z_i^* *as functions of the* x_j *and are expanded as formal power series in the* x_j, *if the equations* (7) *have the form* (12) *the formal power series for* z_1^* *will formally satisfy the partial differential equation* (13), *and the formal power series for* z_2^* *and* z_3^* *will formally satisfy the analogues of* (13) *in* ϕ_2 *and* ϕ_3.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS.